Pairwise Comparisons

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ConQuest 3 is able to fit a logistic pairwise comparison model. This model is known as the Bradley–Terry–Luce (BTL) model (Bradley & Terry, 1952; Luce, 1959)

In the BTL model, we let X_{ij} be a random variable that represents the outcome of the comparison of objects *i* and *j*. If object *i* is judged to have more of an attribute than object *j* then X_{ij} takes the value one and if object *j* is judged to have more of an attribute than object *i* then X_{ij} takes the value zero. Note that ties are not permitted. Under the BTL model we have

$$Pr(X_{ij} = x_{ij}; \delta_i, \delta_j) = \frac{exp\left[x_{ij}\delta_i - (1 - x_{ij})\delta_j\right]}{1 + exp(\delta_i - \delta_j)}$$

where parameter δ_i is a scale location parameter for object *i*. We refer to δ_i as a Thurstonian threshold following Thurstone (1927), although strictly speaking a Thurstonian threshold would be derived using a normal, rather than a logistic comparison model.

ConQuest 3 provides maximum likelihood estimates the BTL model using a Newton-Raphson routine. For the purposes of model identification the mean of the set of location parameters is set to zero.

As the pairwise comparison model does not involve the equivalent of cases any specification of a population model, in particular use of the `regression` command is ignored. Further, any commands that involve case estimates for the latent variables are not available.

To request a pairwise model the option `pairwise` is added to `model` command, further, the `model` command must take the form `model var1-var2!pairwise`.

Pairwise models are restricted in their data layout. The format must include at least two explicit variables in addition to the responses. The two explicit variables given in the model describe the objects that are being compared through the matching set of responses. If the first listed variable in the model statement is judged "better" than the second then a response of one is expected, if the second listed variable in the model statement is judged "better" than a response of zero is expected.

Some examples of possible data layouts are as follows:

Example 1:

```
Format object1 1-6 object2 7-12 response 13;
Model object1-object2 !pairwise;
```

In this case each record of the data file contains the results of one comparison. The result is in column 13 and the codes for the objects being compared are in columns 1-6 and 7-12 respectively.

Example 2:

```
Format object1 1-6 object2 7-12 responses 13-20;
Model object1-object2 !pairwise;
```

In this case each record of the data file contains the results of up to eight comparisons (in columns 13-20). The codes for the objects being compared in all eight of these comparisons are in columns 1-6 and 7-12 respectively. If fewer than eight comparisons are available then blank columns in the range 13-20 would be treated as missing data and ignored.

Example 3:

```
Format obj1 1-2 obj2 3-4 responses 5 obj1 6-7 obj2 8-9 responses 10;
Model obj1-obj2 !pairwise;
```

In this case each record of the data file contains the results of two comparisons (in columns 5 and 10 respectively) for two different pairs of objects. The codes for the objects being compared in the first of these comparisons are in columns 1-2 and 3-4, while the codes for the objects being compared in the second comparison are given in columns 6-7 and 8-9 respectively.

Example 4:

```
Format obj1 1-2 obj2 3-4 responses 5-10 obj2 11-12 responses 13-18;
Model obj1-obj2 !pairwise;
```

In this case each record of the data file contains the results of twelve comparisons (in columns 5-10 and 13-18) for two different pairs of objects. The codes for the objects being compared in the first set of of these comparisons are in columns 1-2 and 3-4, while the codes for the objects being compared in the second comparison are given in columns 1-2 and 11-12. Note how the second use of the same explicit variable name (obj2) means that the second value applies to the second set of responses and since there is only one reference to obj1 it applies to all responses.

Further relevant detail is available from the command reference material that describes the functionality of the format command.

Controlling the Estimation

The `iterations=`*n* and `convergence=`*n* options of the `estimate` command can be used to control the number of iterations and the accuracy of the estimation. The convergence criterion applies to both item and person parameters.

Zeros and Perfects

Finite location parameter estimates are not normally available for objects with zero or perfect scores. To deal with perfect and zero values the user can use the `set` option `zero/perfect=`*n* to set a small value to use as a replacement for zero. In the case of a maximum the small value is subtracted from the maximum possible.

Display of Results

Two output displays are available for the results of pairwise comparisons. The command `Show!table=2`, will provide a list of the parameter estimates and their standard errors. The Wright Map plot will display the item locations graphically. The `order=value` option is available the Wright Maps.

References

Bradley, R.A. and Terry, M.E. (1952). Rank analysis of incomplete block designs, I. the method of paired comparisons. *Biometrika*, 39, 324–345.

Luce, R.D. (1959). *Individual Choice Behaviours*: A Theoretical Analysis. New York: J. Wiley.

Thurstone, L.L. (1927). A law of comparative judgement. *Psychological Review*, 34, 278–286.